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LETTER TO THE EDITOR

Critical dynamics of quantum chains by position space renormalisation group

R B Stinchombe and R R dos Santos†

Department of Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

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Abstract. A decimation transformation is applied to the homogeneous equations of motion for creation and annihilation operators of kink-like excitations of the transverse Ising and XY chains at zero temperature. We obtain the exact dispersion relations and their dynamic scaling form. The static behaviour is derived as the zero frequency limit of the recursion relations.

In this letter we study the dynamic critical behaviour of transverse Ising and isotropic transverse XY chains at zero temperature, by a position space renormalisation group method.

Near a critical point, some spin fluctuation modes (usually the ones with long wavelength) have very long relaxation times (critical slowing down). These set the time scales for time-dependent phenomena near a critical point whose length scales are set by correlation lengths, as in statics. From these ideas, dynamic scaling theories were proposed by Ferrell *et al* (1967) and by Halperin and Hohenberg (1969) which later gave rise to the dynamical renormalisation group (DRG) (for a review, see Hohenberg and Halperin 1977). Until recently, most of the work on DRG relied on ε expansions in momentum space, with the exception of the kinetic Ising model (Glauber 1963) to which the usual position space renormalisation group (PSRG) techniques (Niemeijer and van Leeuwen 1976, Burkhardt and van Leeuwen 1982) have been applied by Marland and Stinchcombe (see Marland 1977), Achiam (1978, 1979), Mazenko *et al* (1978) and Suzuki (1979). The dynamics of the diluted Heisenberg model near the percolation threshold has been recently investigated (Stinchcombe 1983, Harris and Stinchcombe 1983, Stinchcombe and Harris 1983) by a PSRG technique in which the decimation of the spin variables was carried out in the equation of motion for the Green function. This technique had previously been introduced for non-critical dynamics of disordered systems (Gonçalves da Silva and Koiller 1981; see also Marland 1977, Southern *et al* 1983).

The purpose of this letter is to adapt this decimation scheme to a study of quantum dynamical critical behaviour. The system treated is the zero temperature anisotropic transverse XY chain. This includes as special limits the isotropic transverse XY and transverse Ising quantum systems. The Hamiltonian for the general case is (spin $\frac{1}{2}$)

$$\mathscr{H} = -\Gamma \sum_{i=1}^{N} S_i^z - \frac{J}{2} \sum_{i=1}^{N-1} [(1+\eta) S_i^x S_{i+1}^x + (1-\eta) S_i^y S_{i+1}^y] \tag{1}$$

† On leave from Departamento de Física-PUC/RJ, 22453 Rio de Janeiro RJ, Brazil.

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where Γ, J and η are the transverse field, the exchange coupling and the anisotropy parameter, respectively. This Hamiltonian has been diagonalised exactly and correlation functions are also available (Lieb *et al* 1961, Katsura 1962, Niemeijer 1967, Pfeuty 1970, Barouch and McCoy 1971, Vaidya and Tracy 1978), from which it follows that there is a phase transition at $\lambda_c \equiv (J/2\Gamma)_c = 1$ at zero temperature, for any η (see also dos Santos and Stinchcombe 1981).

We will be concerned here with kink-like excitations generated by quantum fluctuations from a state in which the spins are aligned anti-parallel to the transverse field; these fluctuations are the ones which drive the transition. With the aid of the Jordan-Wigner transformation (Lieb *et al* 1961) to fermion operators (c^+ and c^-), the Hamiltonian (1) becomes

$$\mathcal{H} = -\Gamma \sum_{i=1}^{N} c_i^+ c_i^- - \frac{J}{4} \sum_{i=1}^{N-1} [c_i^+ c_{i+1}^- - c_i^- c_{i+1}^+ + \eta(c_i^+ c_{i+1}^+ - c_i^- c_{i+1}^-)] \tag{2}$$

where a constant term has been dropped.

The rate of change of c_n^+, obtained by taking the commutator with the Hamiltonian (2), involves a linear combination of both c^-'s and c^+'s, so in an eigenmode $c_n^+(t)$ has the form

$$c_n^+(t) = v_n^+ \, e^{i\omega t} + \eta u_n \, e^{-i\omega t} \tag{3}$$

and $c_n^-(t)$ is the conjugate of this expression. In place of the site dependent coefficients u_n, v_n it is convenient to use

$$x_n = v_n + \eta u_n \qquad y_n = v_n - \eta u_n. \tag{4}$$

The new variables x, y then satisfy the following equations of motion

$$x_n + \Omega y_n = -\tfrac{1}{2}\lambda[(1-\eta)x_{n-1} + (1+\eta)x_{n+1}] \tag{5}$$

$$y_n + \Omega x_n = -\tfrac{1}{2}\lambda[(1+\eta)y_{n-1} + (1-\eta)y_{n+1}] \tag{6}$$

where the reduced variables $\Omega \equiv \omega/\Gamma$ and $\lambda \equiv J/2\Gamma$ have been introduced.

The dynamic renormalisation group transformation (Marland 1977, Achiam 1978, Stinchcombe 1983) of these variables is obtained by eliminating from the equations of motion the variables x_n, y_n corresponding to sites n of a sublattice of the original system (decimation). This results in equations of motion similar to (5) and (6), where the length scale has been increased by a factor b. For $b = 2$, the resulting equations relate the variables at site n with those at site $n \pm 2$. In the present case this scheme is easily implemented in the two limits $\eta = 1$ (transverse Ising model), and $\eta = 0$ (isotropic transverse XY model). We now discuss these two cases in turn.

(i) *Transverse Ising model ($\eta = 1$)*

Elimination of y from (5) and (6) gives

$$tx_n = x_{n-1} + x_{n+1} \tag{7}$$

where

$$t \equiv [\Omega^2 - (\lambda^2 + 1)]/\lambda \tag{8}$$

is the dynamical scaling variable. Using (7) with n replaced by $n \pm 1$ to relate $x_{n\pm1}$ to $x_{n\pm2}$ and x_n and taking the result back into (7) one gets an equation of similar form to (7) but with $n \pm 1$ replaced on the right-hand side by $n \pm 2$, and on the left-hand side t replaced by

$$t' = t^2 - 2. \tag{9}$$

The change of variables

$$t = 2 \cos k \tag{10}$$

reduces the recursion relation (9) to the simple form

$$k' = 2k \tag{11}$$

so that k transforms as a wavevector under scaling, and is indeed the wavevector apart from a constant factor. Scaling invariance is obtained at the fixed point $t^* = 2$ of (9) or, alternatively, at $k^* = 0$.

Equation (10) yields the dispersion relation

$$\Omega^2 = \lambda^2 + 1 + 2\lambda \cos k \tag{12}$$

which is the exact result (Niemeijer 1967, Pfeuty 1970). By setting $\Omega = 0$ in (12) we see that there is one critical mode for $c^{\pm}$: at $k = 0$ and $\lambda^* = -1$. One should bear in mind that λ^* is negative because in the fermion representation, one looks at excitations from a ground state in which $\Gamma < 0$ and $J = 0$ in (1).

We should note that the existence of a single critical mode (associated with an isolated fixed point) enables us to derive a recursion relation for the static behaviour. Indeed, the dynamic response function (in momentum space) $G(k, \Omega, \lambda)$ transforms as (see, for instance, Ma 1976)

$$G(k, \Omega, \lambda) = b^{2-\eta} G(bk, b^z\Omega, \lambda') \tag{13}$$

under a change of length scale by a factor b, where the coupling constant λ (and λ') is assumed to be near the fixed point λ^*. By setting $\Omega = 0$, $G(k, 0, \lambda)$ is precisely the static correlation function (Ma 1976) whose critical correlations are controlled by λ in the present case. Moreover, the equation of motion satisfied by G is the same as that for $c^{\pm}$, apart from inhomogeneous terms unimportant in the present discussion.

The static scaling is thus the $\Omega = \Omega' = 0$ special case of (9), which gives

$$\lambda' = -\lambda^2 \tag{14}$$

which is equivalent to the exact recursion relation derived by Stinchcombe (1981) for the transverse Ising chain at zero temperature. A solution $\lambda' = -1/\lambda^2$ is also possible, but it is equivalent to matching the disordered phase of the original system into the ordered phase of the scaled system, and *vice versa.*

Near the critical point (i.e. for λ near -1) the usual linearisation procedures applied to the full recursion relation (14), result in

$$\xi \sim |1 + \lambda|^{-\nu}, \qquad \nu = 1. \tag{15}$$

Thus, for large ξ and k small (12) yields the dynamic scaling form (Ferrell *et al* 1967, Halperin and Hohenberg 1969)

$$\Omega = k^z f(k\xi) \tag{16}$$

with

$$z = 1 \tag{17}$$

and

$$f(x)=[1+(c/x)^2]^{1/2} \tag{18}$$

(c is constant). One can then distinguish two regimes

$$\Omega \sim \begin{cases} k & \text{if } k\xi \gg 1 \\ 1/\xi & \text{if } k\xi \ll 1. \end{cases} \tag{19}$$

These results are the two asymptotic forms on either side of the crossover occurring at the place where the wavelength is approximately the correlation length.

The results (16), (17) (but not (18)) can be derived directly from the scaling equation (9) without using its detailed solution (12), by linearising it and its $\Omega=\Omega^*$ special case (14) about the fixed point $(\Omega^*, \lambda^*)=(0,-1)$ (which yields $\Omega'=2\Omega$, $\lambda'-\lambda^*=2(\lambda-\lambda^*)$) and applying standard procedures to these linearised equations.

(ii) *The isotropic transverse XY model ($\eta=0$)*

In this case, (4) implies $x_n \equiv y_n$, so that (5) and (6) become

$$rx_n = x_{n-1}+x_{n+1} \tag{20}$$

where the dynamic variable is now

$$r \equiv -[2(1+\Omega)]/\lambda. \tag{21}$$

The decimation of sites $n\pm 1$ is achieved as before; we get a recursion relation of the same form as (9), but with t now replaced by r. This implies

$$r=2\cos k \tag{22}$$

with k proportional to the wavevector, as before. The dispersion relation then follows as

$$\Omega=|\lambda|\cos k-1 \tag{23}$$

which is the exact result (Niemeijer 1967).

From (23) we see that the critical mode corresponds to

$$k_c=\cos^{-1} 1/|\lambda| \tag{24}$$

so that the transverse XY chain has one critical mode for each value of $|\lambda|>1$, which is not the long wavelength mode; only at $|\lambda|=1$ is the long wavelength mode critical. Thus, the region $|\lambda|\geqslant 1$ corresponds to a critical region (line of fixed points, as suggested by Jullien and Pfeuty 1979), whereas for $|\lambda|<1$ one has a disordered phase.

In view of this, one should have $\lambda'=\lambda=\lambda^*$ in (13), so that setting $\Omega=0$ in (21) would give no information about static behaviour for $|\lambda|>1$. Nevertheless, one can extract the static critical behaviour through this procedure as one approaches $|\lambda|=1$ from the disordered phase. We then obtain

$$\lambda'=\lambda^2/(\lambda^2-2), \qquad |\lambda|\leqslant 1 \tag{25}$$

which has fixed points $\lambda^*=-1$ and 0. The correlation length for $-\lambda\leqslant 1$ is

$$\xi \sim |1+\lambda|^{-\nu}, \qquad \nu=\tfrac{1}{2} \tag{26}$$

so that the dynamic scaling form yields

$$\Omega = k^z g(k\xi) \tag{27}$$

as $|\lambda| \to 1^+$, with

$$z = 2 \tag{28}$$

and

$$g(x) = |1 - 2/x^2|. \tag{29}$$

The above results for ν and z had already been suggested by approximate calculations by Gerber and Beck (1977), Jullien and Pfeuty (1979) and dos Santos and Stinchcombe (1981).

To summarise, we were able to derive an exact dynamical decimation transformation for quantum chains. From this, the dispersion relation for elementary excitations was obtained which satisfies dynamical scaling in the critical region (k, $1/\xi \to 0$). Also, the static recursion relation was derived as a special case (zero frequency) of the dynamical one. Further extensions of the present work will be investigated, including the region $0 < \eta < 1$, and generalisation to non-zero temperature, to higher dimensions, and to random disorder, etc.

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